

A proof for a conjecture of Gyárfás, Lehel, Sárközy and Schelp on Berge-cycles

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Abstract

It has been conjectured that for any fixed $r \geq 2$ and sufficiently large n , there is a monochromatic Hamiltonian Berge-cycle in every $(r-1)$ -coloring of the edges of K_n^r , the complete r -uniform hypergraph on n vertices. In this paper, we give a proof for the case when $r \geq 5$. Together with the previously known results, this gives a positive answer to this conjecture.

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1 Introduction

For a given $r \geq 2$, an r -uniform Berge-cycle of length n , denoted by C_n^r , is an r -uniform hypergraph with the core sequence v_1, v_2, \dots, v_n as the vertices, and distinct edges e_1, e_2, \dots, e_n such that e_i contains v_i, v_{i+1} , where addition is in modulo n . A Berge-cycle of length n in a hypergraph with n vertices is called a *Hamiltonian Berge-cycle*.

For an r -uniform hypergraph H , the *Ramsey number* $R_k(H)$ is the minimum integer n such that in every k -edge coloring of K_n^r there exists a monochromatic copy of H . The existence of such a positive integer is guaranteed by Ramsey's classical result in [9]. Recently, the Ramsey numbers of various variations of cycles in uniform hypergraphs have been studied, e.g. see [5, 6, 8]. In this regard, Gyárfás et al. proposed the following conjecture for Berge-cycles:

Conjecture 1.1. [2] *Assume that $r \geq 2$ is fixed and n is sufficiently large. Then every $(r-1)$ -edge coloring of K_n^r contains a monochromatic Hamiltonian Berge-cycle.*

Conjecture 1.1 states that for a given $r \geq 2$ we have $R_{r-1}(C_n^r) = n$ when n is sufficiently large. The case $r = 2$ is trivial. The case $r = 3$ was proved in [2]. Recently, Maherani and the author gave a proof for the case $r = 4$; see [7]. To see more results on Conjecture

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1.1, we refer the reader to [2, 3, 4] and references therein. In this paper, we show that Conjecture 1.1 holds for all $r \geq 5$. Together with the above results, this gives a positive answer to Conjecture 1.1. The main result is the following theorem:

Theorem 1.2. *Suppose that $r \geq 5$ and $n > 6r \binom{4r}{r-1}$. Then in every $(r-1)$ -edge coloring of K_n^r there is a monochromatic Hamiltonian Berge-cycle.*

2 The proof

Assume that H is an r -uniform hypergraph. The *shadow graph* $\Gamma(H)$ is a graph with vertex set $V(H)$, where two vertices are adjacent if they are covered by at least one edge of H . Consider an $(r-1)$ -edge coloring of $H = K_n^r$ with colors $1, 2, \dots, r-1$ and assume that $G = \Gamma(H)$. For each edge $e = xy$ of G , we assign a list $L(e)$ of colors of all edges of H containing x and y . The color $i \in L(e)$ is *good* for an edge $e \in E(G)$ if at least $r-1$ edges (of H) of color i contain all vertices of e . We consider a new multi-coloring L^* for the edges of G . For each edge $e \in E(G)$, assume that $L^*(e) \subseteq L(e)$ is the set of all good colors for e . Through this note, for each real number x , by $\lfloor x \rfloor$ we mean the largest integer not greater than x and for each natural number m , assume that $[m] = \{1, 2, \dots, m\}$. We will use the following lemma in the proof of Theorem 1.2. We will use the following lemma in the proof of Theorem 1.2.

Lemma 2.1. *[1] Let G be a simple graph and let u and v be nonadjacent vertices in G such that $d_G(u) + d_G(v) \geq n$. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.*

Proof of Theorem 1.2: Suppose to the contrary that there is no monochromatic Hamiltonian Berge-cycle in a given $(r-1)$ -edge coloring c of $H = K_n^r$ with colors $1, 2, \dots, r-1$. For each $1 \leq i \leq r-1$, let W_i be the set of all edges e of $G = \Gamma(H)$ for which $i \notin L^*(e)$. Using Lemma 2.1 in [7], we may assume that the spanning subgraph of G with edge set $E(G) \setminus W_i$ is not Hamiltonian. Now consider $S_i \subseteq W_i$ with minimum cardinality such that the spanning subgraph induced by $E(G) \setminus S_i$ in G is not Hamiltonian. Assume that G_i and G_i^c are the spanning subgraphs of G induced by S_i and $E(G) \setminus S_i$, respectively. For any two non-adjacent vertices x and y of G_i^c the graph $G_i^c + xy$ is Hamiltonian and so by Lemma 2.1 we have $d_{G_i^c}(x) + d_{G_i^c}(y) \leq n-1$. Therefore, for any two adjacent vertices x and y of G_i we have $d_{G_i}(x) + d_{G_i}(y) \geq n-1$. For each color $1 \leq i \leq r-1$, assume that T_i and R_i are the sets of all isolated vertices and all vertices with degree at least $(n-1)/2$ of G_i , respectively. Also, assume that $Q_i = V(G_i) \setminus (T_i \cup R_i)$. Clearly for each $1 \leq i \leq r-1$, Q_i is an independent set in G_i and $R_i \neq \emptyset$. One can easily see that $|R_i| \geq |T_i|$ for each i . Assume to the contrary that for some i we have $|R_i| < |T_i|$. Let $T_i = \{v_1, v_2, \dots, v_{|T_i|}\}$, $R_i = \{w_1, w_2, \dots, w_{|R_i|}\}$ and $Q_i = \{x_1, x_2, \dots, x_{|Q_i|}\}$. Clearly

$$C = v_1 w_1 \dots v_{|R_i|} w_{|R_i|} v_{|R_i|+1} \dots v_{|T_i|} x_1 \dots x_{|Q_i|},$$

is a Hamiltonian cycle in G_i^c , a contradiction. By the same argument we have $|R_i \cup Q_i| > |T_i|$.

For each vertex $x \in V(G)$ and any $1 \leq i \leq r-1$ assume that

$$U_i(x) = \{y \in V(G) \setminus \{x\} \mid i \in L^*(xy)\}, \overline{U}_i(x) = \{y \in V(G) \setminus \{x\} \mid i \notin L^*(xy)\},$$

and $d_i(x)$ is the number of edges of color i containing x in H . For any $I \subseteq [r-1] = \{1, 2, \dots, r-1\}$ set $U_I(x) = \bigcap_{i \in I} U_i(x)$ and $\overline{U}_I(x) = \bigcap_{i \in I} \overline{U}_i(x)$. We say that a set of vertices $S \subseteq V(G)$ *avoids* the set of colors $W \subseteq [r-1]$ if for each $i \in W$ there is a vertex $x \in S$ with $d_i(x) \leq \binom{4r}{r-1}$ or an edge $e = xy$ for $x, y \in S$ with $i \notin L^*(e)$. An argument similar to the proof of Claim 2.3 of Theorem 2.2 in [7] (set $t = 2$ and follow the proof) yields the following result:

Claim 2.2. *Let $P \subseteq [r-1]$ and $|P| = p$. Then there is a set of vertices $Q \subseteq V(G)$ with $|Q| \leq p+1$ such that Q avoids P .*

If there is a subset $S \subseteq V(G)$ that avoids a set of colors containing at least $|S| + 1$ colors $c_1, c_2, \dots, c_{|S|+1}$, then by claim 2.2 there is a subset $S' \subseteq V(G)$ containing at most $r-1-|S|$ vertices that avoids $[r-1] \setminus \{c_1, c_2, \dots, c_{|S|+1}\}$. Now clearly $S \cup S'$ avoids $[r-1]$, which is impossible since the number of edges in H containing $S \cup S'$ is $\binom{n-|S \cup S'|}{r-|S \cup S'|} \geq n-r+1$ and the number of edges of color i (for each $1 \leq i \leq r-1$) containing $S \cup S'$ is at most $\binom{4r}{r-1}$ (note that $n > 6r \binom{4r}{r-1}$). Therefore, each subset $S \subseteq V(G)$ avoids at most $|S|$ colors in $[r-1]$.

Claim 2.3. *For some $0 \leq f \leq r-2$ (by suitable renaming of colors), there are distinct vertices x and $\{y_i\}_{i=1}^{r-1}$ such that $\overline{U}_{r-1}(x) \geq (n-1)/2$, $i \notin L^*(xy_i)$ for any $f+1 \leq i \leq r-1$ and $\{y_i\}_{i=1}^f$ avoids $[f]$.*

Let $S = \{y_i\}_{i=1}^f \subseteq V(G)$ be the largest subset of vertices with $|S| \leq r-1$ that avoids a set containing $|S|$ colors. Note that it is possible to have $S = \emptyset$. Without any loss of generality, we may assume that S avoids $[f]$. The case $f = r-1$ is impossible since the number of edges in H containing S is $n-r+1$ and the number of edges of color i containing S is at most $\binom{4r}{r-1}$. Hence $f \leq r-2$. If $y_i \notin T_j$ for some $1 \leq i \leq f$ and $f+1 \leq j \leq r-1$, then there is a vertex $v \in V(G)$ such that $j \notin L^*(vy_i)$ and so $S \cup \{v\}$ avoids $[f] \cup \{j\}$, a contradiction to the maximality of S . Hence $S \subseteq \bigcap_{i=f+1}^{r-1} T_i$. If $f = r-2$, then set $x \in R_{r-1}$ and $y_{r-1} \in N_{G_{r-1}}(x)$. Since $d_{G_{r-1}}(x) \geq (n-1)/2$, we have $\overline{U}_{r-1}(x) \geq (n-1)/2$ and so there is no thing to prove. Now let $f \leq r-3$. It is easy to see that $\overline{U}_i(x) \cap \overline{U}_j(x) = \emptyset$ for each $x \in V(G)$ and any $f+1 \leq i, j \leq r-1$ with $i \neq j$. Since otherwise, for any $v \in \overline{U}_i(x) \cap \overline{U}_j(x)$, $S \cup \{x, v\}$ avoids $[f] \cup \{i, j\}$, a contradiction. First assume that there is a vertex $x \in \bigcup_{i=f+1}^{r-1} R_i \setminus \bigcup_{i=f+1}^{r-1} T_i$. Without any loss of generality, assume that $x \in R_{r-1}$. Since x has degree at least $(n-1)/2$ in G_{r-1} , we have $\overline{U}_{r-1}(x) \geq (n-1)/2$. On the other hand, for each $i = f+1, \dots, r-1$, we have $x \in R_i \cup Q_i$. Hence for each $f+1 \leq i \leq r-1$, there is a vertex y_i where $xy_i \in E(G_i)$ and so $i \notin L^*(xy_i)$. Now let $\bigcup_{i=f+1}^{r-1} R_i \subseteq \bigcup_{i=f+1}^{r-1} T_i$. We consider the following cases:

Case 1. $R_i \cap R_j = \emptyset$ for any $f+1 \leq i, j \leq r-1$.

Since $|R_i| \geq |T_i|$ and $\bigcup_{i=f+1}^{r-1} R_i \subseteq \bigcup_{i=f+1}^{r-1} T_i$, we have $|R_i| = |T_i|$ for each $f+1 \leq i \leq r-1$ and $\bigcup_{i=f+1}^{r-1} R_i = \bigcup_{i=f+1}^{r-1} T_i$. Clearly for each $f+1 \leq i \leq r-1$, we have $R_i \neq \emptyset$, $Q_i \neq \emptyset$, $d_{G_i}(x) \leq n-1-|T_i|$ when $x \in R_i$ and $d_{G_i}(x) \leq |T_i|$ when $x \in Q_i$. On the other hand, $d_{G_i}(x) + d_{G_i}(y) \geq n-1$ for any two adjacent vertices x and y of G_i . One can easily see that for each $f+1 \leq i \leq r-1$, the bipartite subgraph of G_i with color classes R_i and Q_i is complete and also the subgraph of G_i induced by R_i is complete. First let $f = r-3$. Then $R_{r-1} = T_{r-2}$ and $R_{r-2} = T_{r-1}$. Clearly, there is a vertex

$w \neq y_{r-3}$ such that $r-2 \notin L^*(wy_{r-3})$ (resp. $r-1 \notin L^*(wy_{r-3})$) if $y_{r-3} \in R_{r-2} \cup Q_{r-2}$ (resp. $y_{r-3} \in R_{r-1} \cup Q_{r-1}$). Thus $S \cup \{w\}$ avoids a set containing at least $|S| + 1$ colors, which is impossible by the maximality of S . Now let $f \leq r-4$. Suppose that for every $f+1 \leq i \leq r-2$, we have $|R_{r-1}| \leq |R_i|$, $A_i = R_{r-1} \cap T_i$ and $B_i = R_{r-1} \cap Q_i = R_{r-1} \setminus A_i$. Without any loss of generality, assume that $|A_i| \leq |A_j|$ for $i \leq j$. First let A_{r-3} be non-empty. Clearly $R_i \setminus T_{r-1}$ is non-empty for some $i \in \{r-3, r-2\}$. Therefore, $i, r-1 \notin L^*(uv)$ for all $u \in B_i$ and $v \in R_i \setminus T_{r-1}$ and so $S \cup \{u, v\}$ avoids $[f] \cup \{i, r-1\}$, which is impossible. Now assume that $A_{f+1} = \dots = A_{r-3} = \emptyset$. Thus $R_{r-1} \subseteq T_{r-2}$. If $R_i \setminus T_{r-1}$ is non-empty for some $i \in \{f+1, \dots, r-3\}$, then $i, r-1 \notin L^*(uv)$ for $u \in B_i$ and $v \in R_i \setminus T_{r-1}$ and so $S \cup \{u, v\}$ avoids $[f] \cup \{i, r-1\}$, which is impossible. Otherwise, $f = r-4$ and $R_{r-3} = T_{r-1}$. Since $\bigcup_{i=f+1}^{r-1} R_i = \bigcup_{i=f+1}^{r-1} T_i$, we have $R_{r-1} = T_{r-2}$ and $R_{r-2} = T_{r-3}$. Hence for any three vertices $v_i \in R_i$ where $i = r-3, r-2, r-1$, we have $r-1 \notin L^*(v_{r-2}v_{r-1})$, $r-2 \notin L^*(v_{r-3}v_{r-2})$ and $r-3 \notin L^*(v_{r-3}v_{r-1})$. Therefore, $S \cup \{v_{r-3}, v_{r-2}, v_{r-1}\}$ avoids $[r-1]$, which is impossible.

Case 2. $R_i \cap R_j \neq \emptyset$ for some $f+1 \leq i, j \leq r-1$ and $i \neq j$.

Without any loss of generality, assume that $R_{r-1} \cap R_{r-2} \neq \emptyset$ and let $x \in R_{r-1} \cap R_{r-2}$. If $N_{G_{r-1}}(x) \cap N_{G_{r-2}}(x) \neq \emptyset$, then $r-2, r-1 \notin L^*(xy)$ for any $y \in N_{G_{r-1}}(x) \cap N_{G_{r-2}}(x)$ and so $S \cup \{x, y\}$ avoids a set containing at least $|S| + 2$ colors $1, 2, \dots, f, r-2, r-1$, which is impossible. Therefore, $N_{G_{r-1}}(x) \cap N_{G_{r-2}}(x) = \emptyset$, $V(G) = \{x\} \cup N_{G_{r-1}}(x) \cup N_{G_{r-2}}(x)$ and $d_{G_{r-1}}(x) = d_{G_{r-2}}(x) = (n-1)/2$. First let $f \geq 1$. Let $w = x$ if $y_f \neq x$ and $w \in V(G) \setminus \{x\}$, otherwise. Clearly $r-2 \notin L^*(wy_f)$ or $r-1 \notin L^*(wy_f)$ and so $S \cup \{w\}$ avoids a set containing at least $|S| + 1$ colors, which is impossible by the maximality of S . Therefore, we may assume that $f = 0$. Since $d_{G_{r-1}}(x) = d_{G_{r-2}}(x) = (n-1)/2$, $d_{G_{r-1}}(x) + d_{G_{r-1}}(y) \geq n-1$ for each $y \in N_{G_{r-1}}(x)$ (also, $d_{G_{r-2}}(x) + d_{G_{r-2}}(y) \geq n-1$ for each $y \in N_{G_{r-2}}(x)$) we have $N_{G_i}(x) \subseteq R_i$ for $i = r-2, r-1$. Hence $|R_{r-1}|, |R_{r-2}| \geq (n+1)/2$, $R_{r-1} \cup R_{r-2} = V(G)$ and $\|G_{r-1}\|, \|G_{r-2}\| \geq (n^2-1)/8$. Therefore, for each $1 \leq i \leq r-3$, $R_i \cap R_{r-1} \neq \emptyset$ or $R_i \cap R_{r-2} \neq \emptyset$ and with the same argument we conclude that $\|G_i\| \geq (n^2-1)/8$ for each $1 \leq i \leq r-1$. Since $f = 0$, we have $S_i \cap S_j = \emptyset$ for any $1 \leq i, j \leq r-1$ and so $\|G\| \geq \sum_{i=1}^{r-1} \|G_i\| \geq (n^2-1)/2$, which is impossible.

We choose distinct vertices x and $\{y_i\}_{i=1}^{r-1}$ with desired properties mentioned in Claim 2.3 and maximum f . In the sequel, for simplicity we denote $U_I(x)$ and $\overline{U}_I(x)$ (for $I \subseteq [r-1]$) by U_I and \overline{U}_I , respectively. Since f is maximum, it is easy to see that $\overline{U}_i \cap \overline{U}_j = \emptyset$ for any $f+1 \leq i, j \leq r-1$ and $i \neq j$. Without loss of generality assume that,

$$\overline{U}_{f+1} \subseteq \overline{U}_{f+2} \subseteq \dots \subseteq \overline{U}_{r-1}.$$

Let $Y = \{y_1, y_2, \dots, y_{r-1}\} \setminus \{y_{f+1}\}$. In the rest of our proof we define a Hamiltonian graph Γ with $V(\Gamma) = V(H)$, in such a way that every Hamiltonian cycle C of Γ can be extended to a monochromatic Hamiltonian Berge-cycle of H . For this, we consider the following cases:

Case 1. $f = r-2$.

Consider a graph Γ with vertex set $V(\Gamma) = V(H)$ and edge set $E(\Gamma) = E_1 \cup E_2$, where E_i 's are defined as follows:

- $E_1 = \{uv | u, v \in V(\Gamma) \setminus Y, c(Y \cup \{u, v\}) = r-1\}$. For each $uv \in E_1$, set $e_{uv} = Y \cup \{u, v\}$ and $F_1 = \{e_{uv} | uv \in E_1\}$.
- $E_2 = \{y_i v | 1 \leq i \leq r-2, v \in V(\Gamma) \setminus Y\}$.

For every $u \in V(\Gamma) \setminus Y$, apart from at most $(r-2)\binom{4r}{r-1}$ choices of $v \in V(\Gamma) \setminus (Y \cup \{u\})$ the edges $e_{uv} = Y \cup \{u, v\}$ of H are of color $r-1$. Thus $d_\Gamma(u) \geq n - r\binom{4r}{r-1}$. Also, for each $1 \leq i \leq r-2$ we have $d_\Gamma(y_i) = n - (r-2)$. One can easily see that Dirac's condition implies that the graph Γ is Hamiltonian; see [1].

Now we show that every Hamiltonian cycle in Γ can be extended to a monochromatic Hamiltonian Berge-cycle in H . Suppose that v_1, v_2, \dots, v_n are the vertices of a Hamiltonian cycle C in Γ . For each $i = 1, 2, \dots, n$, we define an edge $f_i \in E(H)$ of color $r-1$ in the same order their subscripts appear so that $\{v_i, v_{i+1}\} \subseteq f_i$ and f_1, f_2, \dots, f_n make a Hamiltonian Berge-cycle with the core sequence v_1, v_2, \dots, v_n . Set $f_i = e_{v_i v_{i+1}} \in F_1$ for $v_i v_{i+1} \in E_1$. Now let $v_i v_{i+1} \in E_2$. Set $f_i = Y \cup \{v_i, v_{i+1}, u_i\}$ of color $r-1$, where $u_i \in V(\Gamma) \setminus (Y \cup \{v_i, v_{i+1}, v_{i+2}\})$ and $f_i \neq f_j$ for every $j < i$. Such an edge exists since at least $n - r\binom{4r}{r-1}$ edges of color $r-1$ contain $Y \cup \{v_i, v_{i+1}\}$, and $\{v_i, v_{i+1}\}$ has been used in at most $2(r-3) + 1$ edges f_j 's for $j < i$.

Case 2. $f \leq r-3$.

Now let

$$M = \{xy_1y_2 \dots y_f u_{f+1} u_{f+2} \dots u_{r-1} | u_i \in \overline{U}_i\}.$$

Clearly, there are at most $(r-2)|\overline{U}_i|$ edges in M of color i for each $f+1 \leq i \leq r-1$. On the other hand, at most $\binom{4r}{r-1}$ edges in M are of color i for every $1 \leq i \leq f$. Therefore,

$$|M| = \prod_{i=f+1}^{r-1} |\overline{U}_i| \leq (r-2) \sum_{i=f+1}^{r-1} |\overline{U}_i| + f \binom{4r}{r-1}.$$

Now let $|\overline{U}_{f+1}| \geq r-1$. Then

$$(r-1)^{r-f-3} (s - (r-f-3)(r-1)) |\overline{U}_{r-1}| \leq |M| \leq (r-2)(s + |\overline{U}_{r-1}|) + f \binom{4r}{r-1},$$

where $s = \sum_{i=f+1}^{r-2} |\overline{U}_i|$. Therefore, $f(s) \leq 0$ where

$$f(x) = ((r-1)^{r-f-3} (x - (r-f-3)(r-1)) - (r-2)) |\overline{U}_{r-1}| - (r-2)x - f \binom{4r}{r-1}.$$

Clearly $f(x)$ is an increasing function, since its derivative is positive for every x . Since $s \geq (r-f-2)(r-1)$, $|\overline{U}_{r-1}| \geq (n-1)/2$ and $n > 6r\binom{4r}{r-1}$, we have

$$f(s) \geq f((r-f-2)(r-1))$$

$$= ((r-1)^{r-f-2} - (r-2)) |\overline{U}_{r-1}| - (r-f-2)(r-2)(r-1) - f \binom{4r}{r-1} > 0,$$

a contradiction. Hence $|\overline{U}_{f+1}| \leq r-2$.

Assume that $Y_i = Y \setminus \{y_i\}$ for every $f+2 \leq i \leq r-1$ and $\overline{U}_{f+1} = \{u_1 (= y_{f+1}), u_2, \dots, u_l\}$. Clearly $l \leq r-2$. Let U be the set of all vertices $y \notin Y \cup \{x\}$, for which the edge $Y \cup \{x, y\}$ is of color $f+1$. It is easy to see that $|U| \geq n - r \binom{4r}{r-1}$. Let U be partitioned into A_1, A_2, \dots, A_{r-1} , where $|A_{r-1}| = \lfloor \frac{n}{2} \rfloor + 1$, $A_{f+1} = \emptyset$ and $|A_i \setminus A_j| \leq 1$ for every $1 \leq i, j \leq r-2$ where $i, j \neq f+1$. Consider a graph Γ with vertex set $V(\Gamma) = V(H)$ and edge set $E(\Gamma) = \bigcup_{i=1}^5 E_i$, where E_i 's are defined as follows:

- Let $E_1 = \{uv | u \in \overline{U}_i \setminus \{y_i\}, f+2 \leq i \leq r-1, v \notin Y \cup \{x, u\}, c(Y_i \cup \{x, u, v\}) = f+1\}$. For each $uv \in E_1$, set $e_{uv} = Y_i \cup \{x, u, v\}$ and $F_1 = \{e_{uv} | uv \in E_1\}$.
- Let $E_2 = \{y_i v | v \in A_i, 1 \leq i \leq r-1\}$ and for each $y_i v \in E_2$, set $e_{y_i v} = Y \cup \{x, v\}$. Also, let $F_2 = \{e_{y_i v} | y_i v \in E_2\}$.
- Let Γ_1 be the graph with vertex set $V(\Gamma)$ and edge set $E_1 \cup E_2$. For each $1 \leq i \leq l$ assume that $\overline{N}_i = Y \cup \overline{U}_{f+1} \cup N_{\Gamma_1}(u_i) \cup \{x\}$ and set $t_i = 0$ if $d_{\Gamma_1}(u_i) > 2r$ and $t_i = 2r+1 - d_{\Gamma_1}(u_i)$, otherwise. Now we show that there are $\sum_{i=1}^l t_i$ distinct edges $e_{ij} \notin F_1 \cup F_2$ (where $1 \leq i \leq l$ and $1 \leq j \leq t_i$) of color $f+1$ with $u_i \in e_{ij}$ such that for each $1 \leq i \leq l$ there exist t_i distinct vertices $v_{ij} \in e_{ij} \setminus \overline{N}_i$. For this, set $r_{11} = 0$, $N_{11} = \overline{N}_1$ and $E_{11} = F_1 \cup F_2$ and follow the following step for $i = 1, 2, \dots, l$.

Step i: For each $1 \leq j \leq t_i$, since $d_{f+1}(u_i) > \binom{4r}{r-1} \geq \binom{|N_{ij}|-1}{r-1} + r_{ij}$, there is an edge $e_{ij} \notin E_{ij}$ of color $f+1$ which contains u_i and a vertex $v_{ij} \in e_{ij} \setminus N_{ij}$. Now set $r_{i(j+1)} = r_{ij} + 1$, $N_{i(j+1)} = N_{ij} \cup \{v_{ij}\}$ and $E_{i(j+1)} = E_{ij} \cup \{e_{ij}\}$ and continue the above procedure. We apply the above procedure t_i times to find the edges e_{ij} and the vertices v_{ij} for $1 \leq j \leq t_i$ with desired properties. Finally let $r_{(i+1)1} = r_{it_i} + 1$, $N_{(i+1)1} = \overline{N}_{i+1}$ and $E_{(i+1)1} = E_{i(t_i+1)}$ and go to Step $i+1$.

Clearly $E_{l(t_l+1)} \setminus E_{11}$ contains $\sum_{i=1}^l t_i$ distinct edges e_{ij} with desired properties. Now set $A = \bigcup_{i=1}^l \bigcup_{j=1}^{t_i} e_{ij}$, $\overline{E}_i = \{u_i v_{ij} | 1 \leq j \leq t_i\}$, $\overline{F}_i = \{e_{ij} | 1 \leq j \leq t_i\}$, $E_3 = \bigcup_{i=1}^l \overline{E}_i$ and $F_3 = \bigcup_{i=1}^l \overline{F}_i$.

- Assume that $U_{12 \dots (r-1)} = \{w_1, w_2, \dots, w_m\}$ and $d_{\Gamma_2}(w_1) \leq d_{\Gamma_2}(w_2) \leq \dots \leq d_{\Gamma_2}(w_m)$, where Γ_2 is the graph with vertex set $V(\Gamma)$ and edge set $\bigcup_{i=1}^3 E_i$. For each $1 \leq i \leq r$ if $d_{\Gamma_2}(w_i) > 2r$, then set $t'_i = 0$. Otherwise, set $t'_i = 2r+1 - d_{\Gamma_2}(w_i)$. Also, set $N'_i = Y \cup \overline{U}_{f+1} \cup N_{\Gamma_2}(w_i) \cup \{x\}$. An argument similar to one used in the definition of E_3 shows that there are $\sum_{i=1}^r t'_i$ distinct edges $e'_{ij} \notin F_1 \cup F_2 \cup F_3$ (where $1 \leq i \leq r$ and $1 \leq j \leq t'_i$) of color $f+1$ with $w_i \in e'_{ij}$ such that for each $1 \leq i \leq r$ there exist t'_i distinct vertices $v'_{ij} \in e'_{ij} \setminus N'_i$. Now, set $B = \bigcup_{i=1}^r \bigcup_{j=1}^{t'_i} e'_{ij}$, $E'_i = \{w_i v'_{ij} | 1 \leq j \leq t'_i\}$, $F'_i = \{e'_{ij} | 1 \leq j \leq t'_i\}$, $E_4 = \bigcup_{i=1}^r E'_i$ and $F_4 = \bigcup_{i=1}^r F'_i$.
- $E_5 = \{xv | v \in V(\Gamma) \setminus (Y \cup \overline{U}_{f+1} \cup A \cup B)\}$.

Claim 2.4. *The graph Γ is Hamiltonian.*

Assume that $d_1 \leq d_2 \leq \dots \leq d_n$ are degrees of the vertices of Γ . We show that $d_1 > 2r$ and $d_{n-i} \geq n-i$ for each $2r-1 \leq i \leq \frac{n}{2}$. Therefore, Chvátal's condition implies the existence of a Hamiltonian cycle in Γ . Clearly, $d_\Gamma(x) \geq n-4r^3$. Let D_i be the set of all edges of color i containing the vertices of $Y_{r-1} \cup \{x\}$ for each $i \neq f+1, r-1$ and let

$$W = \bigcup_{i \neq f+1, r-1} \bigcup_{e \in D_i} (e \setminus (Y_{r-1} \cup \{x\})).$$

Clearly $|D_i| \leq \binom{4r}{r-1}$ and so $|W| \leq 2(r-3)\binom{4r}{r-1}$. For every $u \in \overline{U}_{r-1} \setminus (W \cup \{y_{r-1}\})$, apart from at most $r-2$ choices of $v \in V(\Gamma) \setminus (Y \cup \{x, u\})$, we have $uv \in E_1$. So $d_\Gamma(u) > n-2r$, where $u \in \overline{U}_{r-1} \setminus (W \cup \{y_{r-1}\})$. On the other hand, $|\overline{U}_{r-1} \setminus (W \cup \{y_{r-1}\})| \geq \frac{n-3}{2} - 2(r-3)\binom{4r}{r-1}$ and so $d_i > n-2r$ for $i \geq \lfloor (n+5)/2 \rfloor + 2(r-3)\binom{4r}{r-1}$. Because of $n > 6r\binom{4r}{r-1}$, we have $d_i > n-2r$ for $i \geq n - r\binom{4r}{r-1}$. On the other hand, for every $u \in \overline{U}_{r-1} \cap W \setminus \{y_{r-1}\}$ (also, $u \in \overline{U}_i \setminus \{y_i\}$ for each $f+2 \leq i \leq r-2$), apart from at most $(r-2)\binom{4r}{r-1}$ choices of $v \in V(\Gamma) \setminus (Y \cup \{x, u\})$ we have $uv \in E_1$ and so $d_\Gamma(u) > n - r\binom{4r}{r-1}$. Hence $d_i > n - r\binom{4r}{r-1}$ for $i \geq \lfloor (n+3)/2 \rfloor$. Since $y_i v \in E(\Gamma)$ for each $v \in A_i$, we have $d_\Gamma(y_{r-1}) > n/2$ and $d_\Gamma(y_i) > 2r$ for each $1 \leq i \leq r-1$ and $i \neq f+1$. Hence $d_i > n/2$ for $i \geq \lfloor (n+1)/2 \rfloor$. Moreover, for every $u_i \in \overline{U}_{f+1}$, we have $d_\Gamma(u_i) \geq d_{\Gamma_1}(u_i) + t_i > 2r$.

Now let $U_{12\dots(r-1)} = \{w_1, w_2, \dots, w_m\} \neq \emptyset$ with $d_{\Gamma_2}(w_1) \leq d_{\Gamma_2}(w_2) \leq \dots \leq d_{\Gamma_2}(w_m)$, where Γ_2 is the graph with vertex set $V(\Gamma)$ and edge set $\bigcup_{i=1}^3 E_i$ and $|\overline{U}_{r-1} \setminus \{y_{r-1}\}| = k$. We show that $d_\Gamma(w_{r+1}) \geq d_{\Gamma_2}(w_{r+1}) > 2r$ and so by the definition of E_4 we conclude that

$$\min\{d_\Gamma(w_i) | 1 \leq i \leq m\} > 2r.$$

For $i = 1, \dots, m$, consider

$$N_i = \{\{x, y_1, y_2, \dots, y_{r-2}, v, w_i\} \setminus \{y_{f+1}\} \mid v \in \overline{U}_{r-1} \setminus \{y_{r-1}\}\}.$$

For every $1 \leq i \leq m$, suppose that n_i is the number of edges of color $f+1$ in N_i . Clearly $d_{\Gamma_2}(w_i) \geq n_i$. Among all mk edges in $\bigcup_{i=1}^m N_i$ there are at most $\binom{4r}{r-1}$ edges of color i for $i \neq f+1, r-1$ and $(r-2)k$ edges of color $r-1$. Hence $\sum_{i=1}^m n_i \geq (m-r+2)k - (r-3)\binom{4r}{r-1}$. If $d_{\Gamma_2}(w_{r+1}) \leq 2r$, then $\sum_{i=1}^{r+1} n_i \leq \sum_{i=1}^{r+1} d_{\Gamma_2}(w_i) \leq 2r(r+1)$. Therefore

$$\sum_{i=r+2}^m n_i \geq (m-r+2)k - (r-3)\binom{4r}{r-1} - 2r(r+1) > (m-r+1)k,$$

which is impossible since $|\bigcup_{i=r+2}^m N_i| = (m-r-1)k$. Thus $d_\Gamma(w_{r+1}) \geq d_{\Gamma_2}(w_{r+1}) > 2r$ and consequently $d_\Gamma(w_i) > 2r$ for $r+1 \leq i \leq m$. On the other hand, according to the definition of Γ , we have $d_\Gamma(w_i) \geq d_{\Gamma_2}(w_i) + t'_i > 2r$ for each $1 \leq i \leq r$ and so $\min\{d_\Gamma(w_i) | 1 \leq i \leq m\} > 2r$.

Based on the previous discussions, $d_1 > 2r$ and $d_{n-i} \geq n-i$ for each $2r-1 \leq i \leq \frac{n}{2}$. Now, Chvátal's condition implies the existence of a Hamiltonian cycle in Γ .

Claim 2.5. *There is a monochromatic Hamiltonian Berge-cycle of color $f+1$ in H .*

We show that every Hamiltonian cycle in Γ can be extended to a monochromatic Hamiltonian Berge-cycle of color $f+1$ in H . Suppose that $v_1, v_2, \dots, v_n = x$ are the vertices of a Hamiltonian cycle C in Γ . Now for each $i = 1, 2, \dots, n$, we define an edge $f_i \in E(H)$ of color $f+1$ in the same order their subscripts appear so that $\{v_i, v_{i+1}\} \subseteq f_i$ and f_1, f_2, \dots, f_n make a Hamiltonian Berge-cycle with the core sequence v_1, v_2, \dots, v_n . First, let $1 \leq i \leq n-2$. If $v_i v_{i+1} \in E_j$ for some $j \in \{1, 2\}$, set $f_i = e_{v_i v_{i+1}} \in F_j$. Set $f_i = e_{kj} \in F_3$ for $\{v_i, v_{i+1}\} = \{u_k, v_{kj}\}$ and $u_k v_{kj} \in E_3$ where $k \in \{1, 2, \dots, l\}$ and $1 \leq j \leq t_k$. Also, set $f_i = e'_{kj} \in F_4$ for $\{v_i, v_{i+1}\} = \{w_k, v'_{kj}\}$ and $w_k v'_{kj} \in E_4$ where $k \in \{1, 2, \dots, r\}$ and $1 \leq j \leq t'_k$. Now let $i = n-1$. Since $\{v_{n-1}, x\}$ has been used in at most one of the edges f_i 's, where $1 \leq i \leq n-2$ (only possibly in f_{n-2}) and

$f + 1 \in L^*(v_{n-1}x)$, then we can choose an appropriate edge f_{n-1} of color $f + 1$ where $f_{n-1} \neq f_i$ for each $1 \leq i \leq n - 2$. Similarly, for $i = n$, since $\{x, v_1\}$ has been used in at most one of the edges f_i 's, where $1 \leq i \leq n - 1$ (only possibly in f_1) and $f + 1 \in L^*(xv_1)$, then we can choose an appropriate edge f_n of color $f + 1$. ■

References

- [1] J. A. Bondy, U. S. R. Murty, Graph theory with applications, American Elsevier, New York, 1976.
- [2] A. Gyárfás, J. Lehel, G.N. Sárközy and R.H. Schelp, Monochromatic Hamiltonian Berge-cycles in colored complete uniform hypergraphs, *J. Combin. Theory Ser. B.* **98** (2008), 342–358.
- [3] A. Gyárfás, G.N. Sárközy and E. Szemerédi, Long monochromatic Berge-cycles in colored 4-uniform hypergraphs, *Graphs Combin.* **26** (2010), 71–76.
- [4] A. Gyárfás, G.N. Sárközy and E. Szemerédi, Monochromatic matchings in the shadow graph of almost complete hypergraphs, *Ann. Combin.* **14** (2010), 245–249.
- [5] P. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński, M. Simonovits and J. Skokan, The Ramsey number for hypergraph cycles I, *J. Combin. Theory Ser. A* **113** (2006), 67–83.
- [6] P. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński and J. Skokan, The Ramsey number for 3-uniform tight hypergraph cycles, *Combin. Probab. Comput.* **18** (2009), 165–203.
- [7] L. Maherani and G.R. Omid, Monochromatic Hamiltonian Berge-cycles in colored hypergraphs, <http://arxiv.org/pdf/1403.2894v1.pdf>.
- [8] G.R. Omid and M. Shahsiah, Ramsey numbers of 3-uniform loose paths and loose cycles, *J. Combin. Theory Ser. A*, **121** (2014), 64–73.
- [9] F.P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc. Ser. 2* **30** (1930), 264–286.